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STABILITY AND VIBRATION CHARACTERISTICS OF AXIALLY MOVING PLATES

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Abstract—Stability and vibration characteristics of two dimensional axially moving plates have been investigated. The closed form solution of the speed at the onset of instability is predicted by linear plate theory and exact boundary conditions. The speed at the onset of instability is the lowest speed at which non-trivial equilibrium position exists (static analysis) or the lowest speed at which the real part of one eigenvalue impends to be non-zero (dynamic analysis). The critical speed is the speed at which the transport speed of the plate equals the propagation speed of a transverse wave in the plate. The results show that the critical speed equals the speed at the onset of instability predicted by static and dynamic analyses. The speed at the onset of instability increases as the ratio of the length to the width of the plate decreases and as the flexural stiffness of the plate increases. One dimensional beam theory always overestimates the speed at the onset of instability and string theory always underestimates that speed. The plate may experience divergent or flutter instability at supercritical transport speed. A second stable region above the critical speed may exist for plates with slenderness ratio greater than a critical value determined by the stiffness ratio and Poisson's ratio. This opens the possibility of stable operation at speeds greater than the critical speed. © 1997 Elsevier Science Ltd.

INTRODUCTION

Axially moving materials are found in industry as band saw blades, magnetic tapes, paper webs, plastic sheets, films, transmission cables, and the like. Above a critical speed, the axially moving material experiences divergent or flutter instability. The instability of band saw blades leads to loss of raw material, low surface quality and unsatisfactory cutting performance. Excessive vibration of a computer tape degrades the signal and can cause improper data storage. Flutter of a paper web degrades quality, increases defects, and can lead to breakage of the web. To ensure that the systems are under stable operation, the occurrence of instability must be predicted and controlled.

Traveling threadline theory (Archibald and Emsile, 1958; Sack, 1954) can be applied to predict the instability of the materials with small flexural stiffness. One dimensional beam theory (Simpson, 1973) is useful when the effects of the boundary conditions at the width ends on the prediction are negligible.

Gorman (1982) investigated the free vibration of stationary rectangular plates. Lin and Mote (1995) investigated the equilibrium displacement and stress distribution of a two dimensional axially moving web under transverse loading. It is followed by a paper (Lin and Mote, 1996) predicting the wrinkling instability and the corresponding wrinkled shape of a web with small flexural stiffness. However, the stability and the vibration characteristics of axially moving plates have not been fully understood.

The purpose of this paper is to investigate the stability and the vibration characteristics of two dimensional axially moving plates with two simply supported and two free edges, subjected to uniform in-plane tension in the transport direction. The closed form solution of the speed at the onset of instability is predicted by linear plate theory and exact boundary conditions. The axially moving beam and string theories are also considered for comparison. The instability of the plate is predicted by determining the existence of non-trivial equilibrium position (static analysis) and the study of an eigenvalue problem (dynamic analysis). The speed at the onset of instability is the lowest speed at which the non-trivial equilibrium position exists or the lowest speed at which the real part of one eigenvalue impends to be non-zero. The critical speed is the speed at which the transport speed of the plate equals the propagation speed of a transverse wave in the plate. The results show that the critical speed equals the speed at the onset of instability predicted by static and dynamic analyses. The speed at the onset of instability increases as the ratio of the length to the width of the plate decreases and as the flexural stiffness of the plate increases. One dimensional beam theory always overestimates the speed at the onset of instability and string theory always underestimates that speed. The plate may experience divergent or flutter instability at supercritical transport speed. A second stable region above the critical speed may exist for plates with slenderness ratio greater than a critical value determined by the stiffness ratio and Poisson's ratio. This opens the possibility of stable operation at speeds greater than the critical speed.

EQUATION OF MOTION

The equation governing transverse motion of the two dimensional axially moving plate, in coordinates (x, y) fixed in space, in free vibration in Fig. 1 is

$$\rho(\hat{w}_{,\hat{t}\hat{t}}+2v\hat{w}_{,\hat{x}\hat{t}}+v^2\hat{w}_{,\hat{x}\hat{x}})-T\hat{w}_{,\hat{x}\hat{x}}+D\hat{\nabla}^4\hat{w}=0, \tag{1}$$

where a comma denotes partial differentiation and $\hat{\nabla}^4\hat{w} = \hat{w}_{,\hat{x}\hat{x}\hat{x}\hat{x}}+2\hat{w}_{,\hat{x}\hat{x}\hat{y}\hat{y}}+\hat{w}_{,\hat{y}\hat{y}\hat{y}\hat{y}}$. The boundary conditions at the free edges, $\hat{y} = 0$ and $\hat{y} = B$, are

$$[\hat{w}_{,\hat{y}\hat{y}\hat{y}}+(2-v)\hat{w}_{,\hat{x}\hat{x}\hat{y}}] = 0 \tag{2}$$

$$[\hat{w}_{,\hat{y}\hat{y}}+v\hat{w}_{,\hat{x}\hat{x}}] = 0 \tag{3}$$

and at the simply supported edges, $\hat{x} = 0$ and $\hat{x} = L$, are

$$\hat{w} = 0 \tag{4}$$

$$[\hat{w}_{,\hat{x}\hat{x}}+v\hat{w}_{,\hat{y}\hat{y}}] = 0. \tag{5}$$

(5) can be simplified as

$$\hat{w}_{,\hat{x}\hat{x}} = 0 \tag{5a}$$

because $\hat{w}_{,\hat{y}\hat{y}}$ is zero along the simply supported edges. Here, $\hat{w}(\hat{x}, \hat{y}, \hat{t})$ denotes the transverse displacement of the plate at $(\hat{x}, \hat{y})$ and time $\hat{t}$; $D = Eh^3/[12(1-v^2)]$; E is Young's modulus;

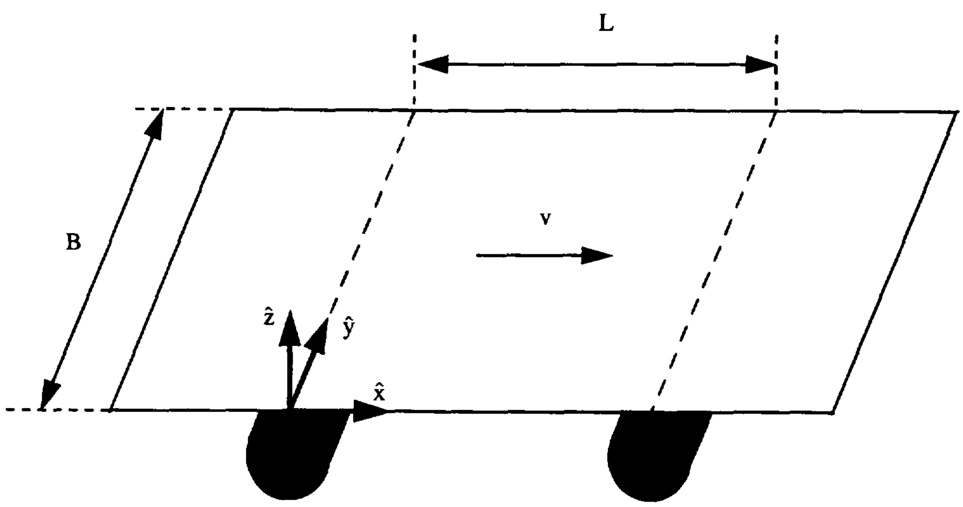

Fig. 1. Axially moving plate.

ρ is mass per area; v is transport speed; ν is Poisson's ratio; h is thickness; T is the applied longitudinal tension. The material properties are uniform and constant. Let the dimensionless parameters be

$$w = \frac{\hat{w}}{h}; \quad t = \hat{t}\left(\frac{T}{\rho L^2}\right)^{1/2}; \tag{6a}$$

$$x = \frac{\hat{x}}{L}; \quad y = \frac{\hat{y}}{L}. \tag{6b}$$

Define

$$\varepsilon = \frac{D}{TL^2}; \quad C = \frac{v}{\left(\frac{T}{\rho}\right)^{1/2}}; \tag{7a}$$

$$\xi = \frac{L}{B}; \quad S = \frac{C^2 - 1}{\varepsilon}. \tag{7b}$$

The stiffness ratio, ε, is a non-dimensional ratio of the flexural stiffness to the stiffness derived from the applied longitudinal tension. C is the constant, non-negative speed ratio of the transport speed to the propagation speed of a transverse wave in a string. The slenderness ratio ξ is the ratio of the length to the width of the plate. S is introduced to simplify the presentation of the equation of motion. Substitution of (6) and (7) into (1) gives the dimensionless equation of motion

$$w_{,tt} + 2Cw_{,xt} + (C^2 - 1)w_{,xx} + \varepsilon\nabla^4 w = 0, \tag{8}$$

where ∇^4 is the dimensionless biharmonic operator.

STATIC ANALYSIS

The axially moving plate can be unstable if multiple equilibrium positions exist at any problem specification. The speed at the onset of instability is the lowest speed at which multiple equilibrium positions exist.

The equilibrium position in (8) satisfy

$$Sw_{,xx} + \nabla^4 w = 0. \tag{9}$$

We can conclude that non-trivial equilibrium position does not exist for $S < 0$ by comparing eqn (9) with the equation of motion of a stationary plate subjected to in-plane compression or tension in the longitudinal direction (Timoshenko, 1936). Thus, the necessary condition for the existence of multiple equilibrium positions of an axially moving plate is

$$S \geqslant 0 \tag{10a}$$

or

$$C \geqslant 1. \tag{10b}$$

Non-trivial equilibrium position w_{mk} $(m, k = 1, 2, \ldots)$ exists when $S = S_{mk}$. The value S_{mk}, depending on the slenderness ratio ξ, can be determined from the transcendental

equation in the Appendix. There are two sets of non-trivial equilibrium positions:
(i) For $0 \leqslant S \leqslant (m\pi)^2$

$$w_{m1} = \sin(m\pi x)\{[A_{m1}\sinh(\alpha_m y) + \bar{A}_{m1}\cosh(\alpha_m y)] + [\tilde{A}_{m1}\sinh(\beta_m y) + \hat{A}_{m1}\cosh(\beta_m y)]\}. \quad (11a)$$

(ii) For $S > (m\pi)^2$

$$w_{mn} = \sin(m\pi x)\{[A_{mn}\sinh(\alpha_m y) + \bar{A}_{mn}\cosh(\alpha_m y)] + [\tilde{A}_{mn}\sin(\gamma_m y) + \hat{A}_{mn}\cos(\gamma_m y)]\}, \quad (11b)$$

where

$$\alpha_m = (m^2\pi^2 + m\pi S^{1/2})^{1/2} \quad (12a)$$

$$\beta_m = (m^2\pi^2 - m\pi S^{1/2})^{1/2} \quad (12b)$$

$$\gamma_m = (-m^2\pi^2 + m\pi S^{1/2})^{1/2}; \quad (12c)$$

A_{mk}, $\bar{A}_{mk}$, $\tilde{A}_{mk}$, and $\hat{A}_{mk}$ are constants; and $n = 2, 3, \ldots$. Let S^* be the smallest S_{mk}. Thus, the speed at the onset of instability, C^*, of an axially moving plate is

$$C^* = \sqrt{1 + \varepsilon S^*}. \quad (13)$$

DYNAMIC ANALYSIS

Wickert's approach (1990) using beam theory is applied to derive the equation of motion for plate in matrix form. Define the differential operators

$$M = I; \quad G = 2C\frac{\partial}{\partial x}; \quad K = -(1 - C^2)\frac{\partial^2}{\partial x^2} + \varepsilon\nabla^4. \quad (14)$$

Equation (8) becomes

$$Mw_{,tt} + Gw_{,t} + Kw = 0. \quad (15)$$

Let

$$w(x, y, t) = u(x, y)\,e^{\lambda t}, \quad (16)$$

where λ is a complex number. Substitution of (16) into (15) gives

$$\lambda^2 Mu + \lambda Gu + Ku = 0. \quad (17)$$

In terms of the vector and matrix operators

$$\tilde{\mathbf{u}} = \begin{Bmatrix} \lambda u \\ u \end{Bmatrix}; \quad \mathbf{A} = \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} G & K \\ -K & 0 \end{bmatrix}, \quad (18)$$

the equation of motion (17) becomes

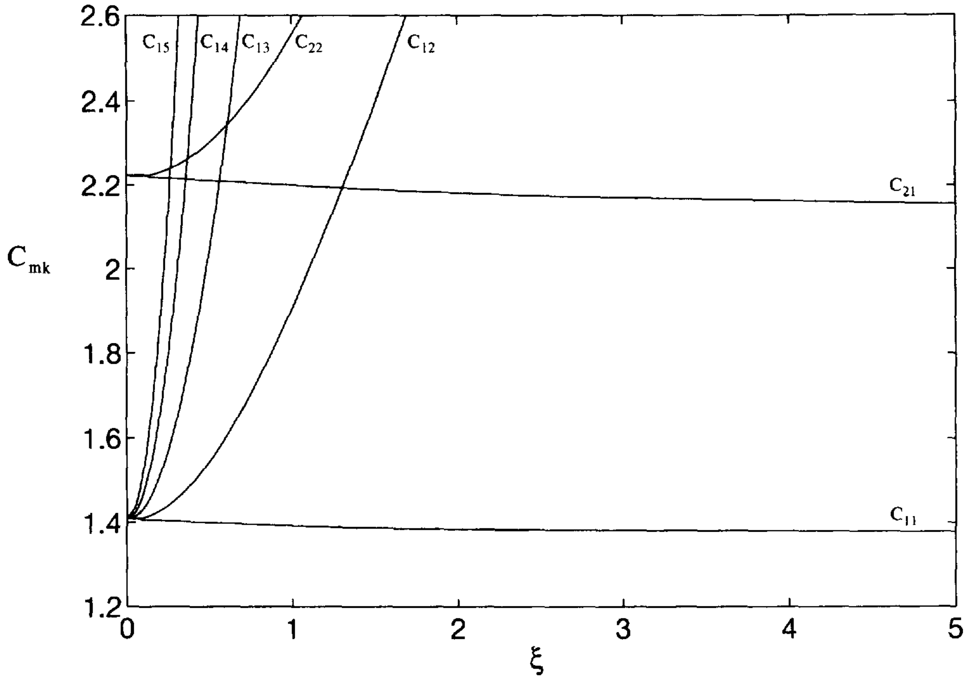

Fig. 2. The speed C_{mk} at which non-trivial equilibrium position w_{mk} exists, with $\varepsilon = 0.1$ and $\nu = 0.3$.

$$\lambda \mathbf{A}\tilde{\mathbf{u}} + \mathbf{B}\tilde{\mathbf{u}} = 0. \tag{19}$$

Matrix differential operator **A** is symmetric and **B** is skew-symmetric with the inner product of two vectors $\tilde{\mathbf{u}}_1$ and $\tilde{\mathbf{u}}_2$ defined as

$$\langle \tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2 \rangle = \int_0^{1/\xi} \int_0^1 \tilde{\mathbf{u}}_1^{\mathrm{T}} \bar{\tilde{\mathbf{u}}}_2 \, \mathrm{d}x \, \mathrm{d}y, \tag{20}$$

where the overbar denotes complex conjugation. The eigenvalue problem, formed by (19) plus appropriate boundary conditions, is solved using the Galerkin method with comparison functions w_{mk} in (11). The speed at the onset of instability is the speed at which the smallest natural frequency vanishes and the real part of the corresponding eigenvalue impends to be non-zero.

RESULTS AND DISCUSSION

The speed at the onset of instability, C^*, of an axially moving plate is the lowest transport speed at which multiple equilibrium positions exist (static analysis) or the lowest speed at which the real part of one eigenvalue in (16) impends to be non-zero (dynamic analysis). Let σ and ω denote the real and imaginary parts of the eigenvalue λ in (16), respectively. Non-zero σ indicates the instability of the system and ω is the circular natural frequency of the axially moving plate. The plate can experience divergent instability (one mode with non-zero σ and zero ω) or flutter instability (one mode with non-zero σ and non-zero ω) when it transports at a speed greater than the speed at the onset of instability.

Let C_{mk} $(m, k = 1, 2, \ldots)$ be the transport speed at which non-trivial equilibrium position w_{mk} in (11a) and (11b) exists. Figure 2 shows C_{mk} with $\varepsilon = 0.1$ and $\nu = 0.3$. w_{11} is the dominant instability mode because $C_{11} < C_{pq}$ for $(p, q) \neq (1, 1)$. Therefore, the speed at the onset of instability C^* is the speed at which w_{11} exists,

$$C^* = C_{11}. \tag{21}$$

C_{m1} decreases and C_{mn} ($n = 2, 3, \ldots$) increases as the slenderness ratio ξ increases. The speed at the onset of instability, C^*, decreases as the slenderness ratio increases.

For plates with small slenderness ratio ($\xi < 0.2$), many different w_{mk} may occur in a small range of transport speed. For instance, the speed range $1.4 < C < 1.45$ with $\varepsilon = 0.1$ and $\nu = 0.3$. In this case, the occurrence of jumps between modes may be frequently observed.

The speed at the onset of instability of an axially moving string is

$$C_s^* = 1 \tag{22}$$

and that of an axially moving beam is

$$C_b^* = \sqrt{1+\varepsilon\pi^2}. \tag{23}$$

Figure 3 shows the speed at the onset of instability, C^* in (13), for plates with different slenderness ratios $\xi = 0.5$, 1 and 10. C^* increases as the stiffness ratio ε increases. When $\varepsilon = 0$, C^* in (13) predicted by plate theory is equivalent to C_s^* in (22) predicted by string theory. For an infinitely wide plate ($\xi \to 0$), the effects of the free end boundary conditions on C^* are negligible and C^* predicted by the plate theory is equivalent to C_b^* predicted by the beam theory. For a plate with finite width, one dimensional beam theory always overestimates C^* and string theory always underestimates C^*. The error of the speed at the onset of instability predicted by the beam theory is significant for materials with large stiffness ratio and large slenderness ratio. For instance, with $\varepsilon = 1.0$ and $\xi = 10$, the one dimensional beam theory overestimates C^* by 4.30%. But, with $\varepsilon = 0.1$ and $\xi = 0.5$, the beam theory overestimates C^* by only 0.67%.

The pertinent specifications of one paper product are $\varepsilon = 1.58\times10^{-4}$, and $\xi = 2$ ($E = 5\times10^9$ N/m^2, $T = 55$ N/m, $\nu = 0.3$, $L = 1.194$ m, $B = 0.597$ m, $h = 0.3$ mm). In this case, the differences between C^* predicted by string, beam, and plate theories are negligible because the stiffness ratio is small. A high speed band saw can have the following specifications: $\varepsilon = 1.127$, and $\xi = 4$ ($E = 2\times10^{11}$ N/m^2, $T = 4\times10^4$ N/m, $\nu = 0.3$, $L = 1.0$ m,

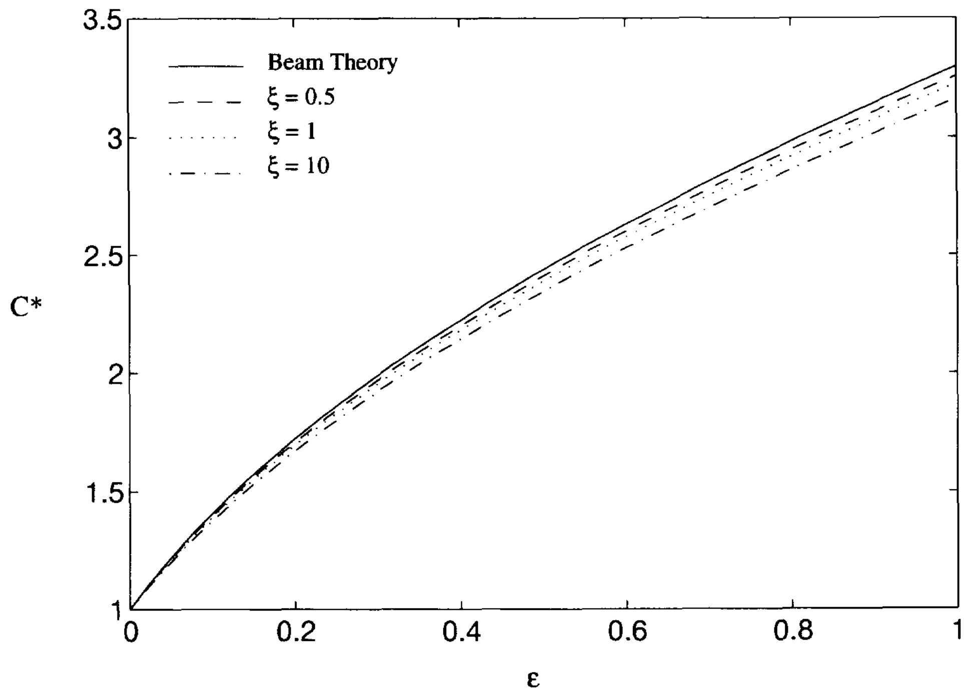

Fig. 3. The speed at the onset of instability, C^*, with different slenderness ratios $\xi = 0.5$, 1, and 10.

$B = 0.25$ m, $h = 0.0135$ m). In this case, the beam theory overestimates C^* by 4.1%. C^* predicted by the plate theory is useful for high stiffness, wide band saw blades.

Figure 4 shows the non-trivial equilibrium positions of the axially moving plate with $\xi = 1.0$, and $\nu = 0.3$. The deflections of w_{11} in Fig. 4(a) and w_{21} in Fig. 4(b) are described by a sine function in the longitudinal direction and a function including two sets of hyperbolic sines and hyperbolic cosines in the width direction (eqn (11a)). The variations of the deflections of w_{11} and w_{21} in the width direction are negligible except near the free edges. The deflections of w_{12} in Fig. 4(c) and w_{22} in Fig. 4(d) are described by a sine function in the longitudinal direction and a function including one set of sine, cosine, hyperbolic sine, and hyperbolic cosine in the width direction (eqn (11b)). The slopes of the deflections

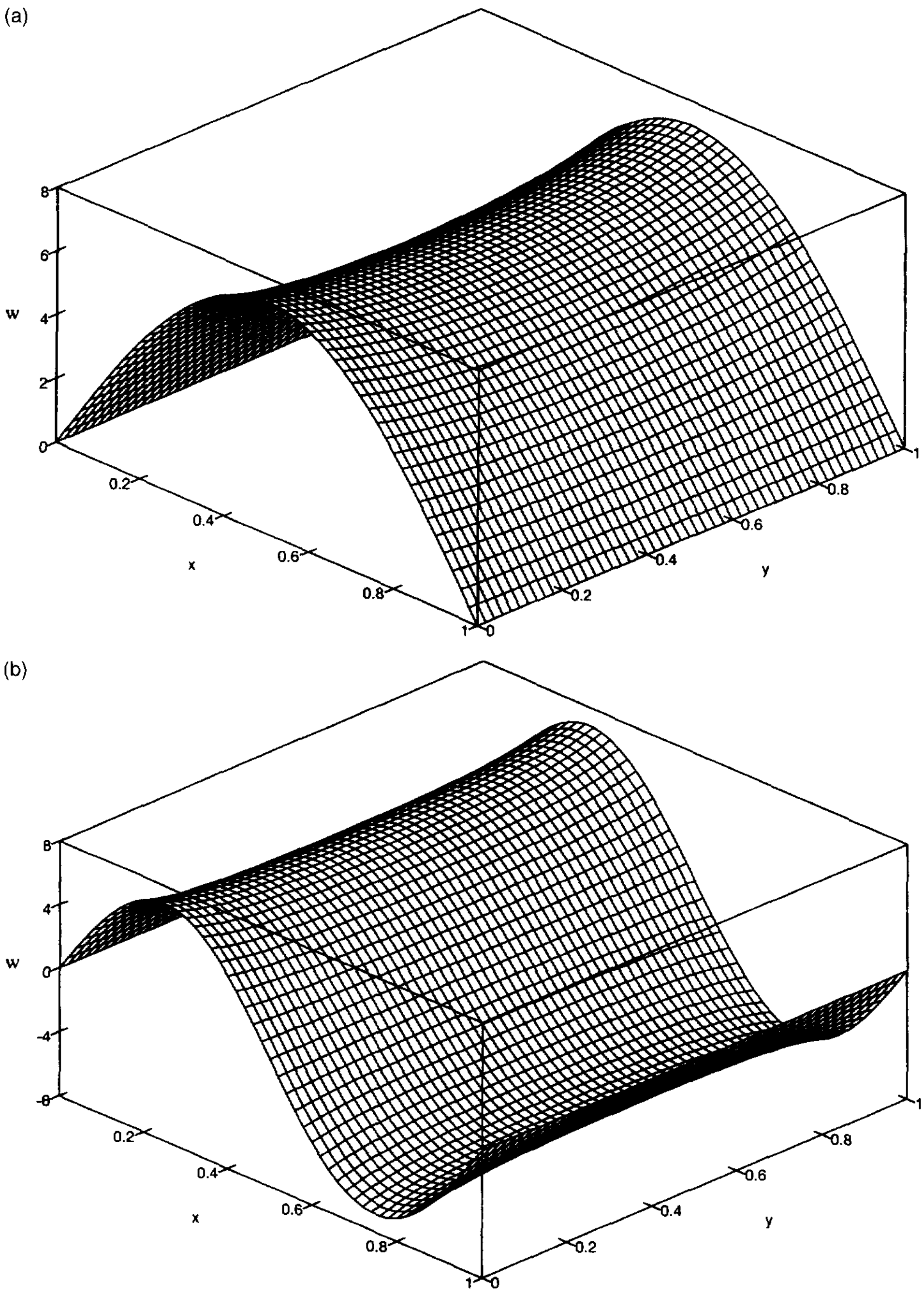

Fig. 4. The non-trivial equilibrium positions of the axially moving plate with $\xi = 1.0$ and $\nu = 0.3$. (a) w_{11}, (b) w_{21}, (c) w_{12}, (d) w_{22}. (*Continued overleaf.*)

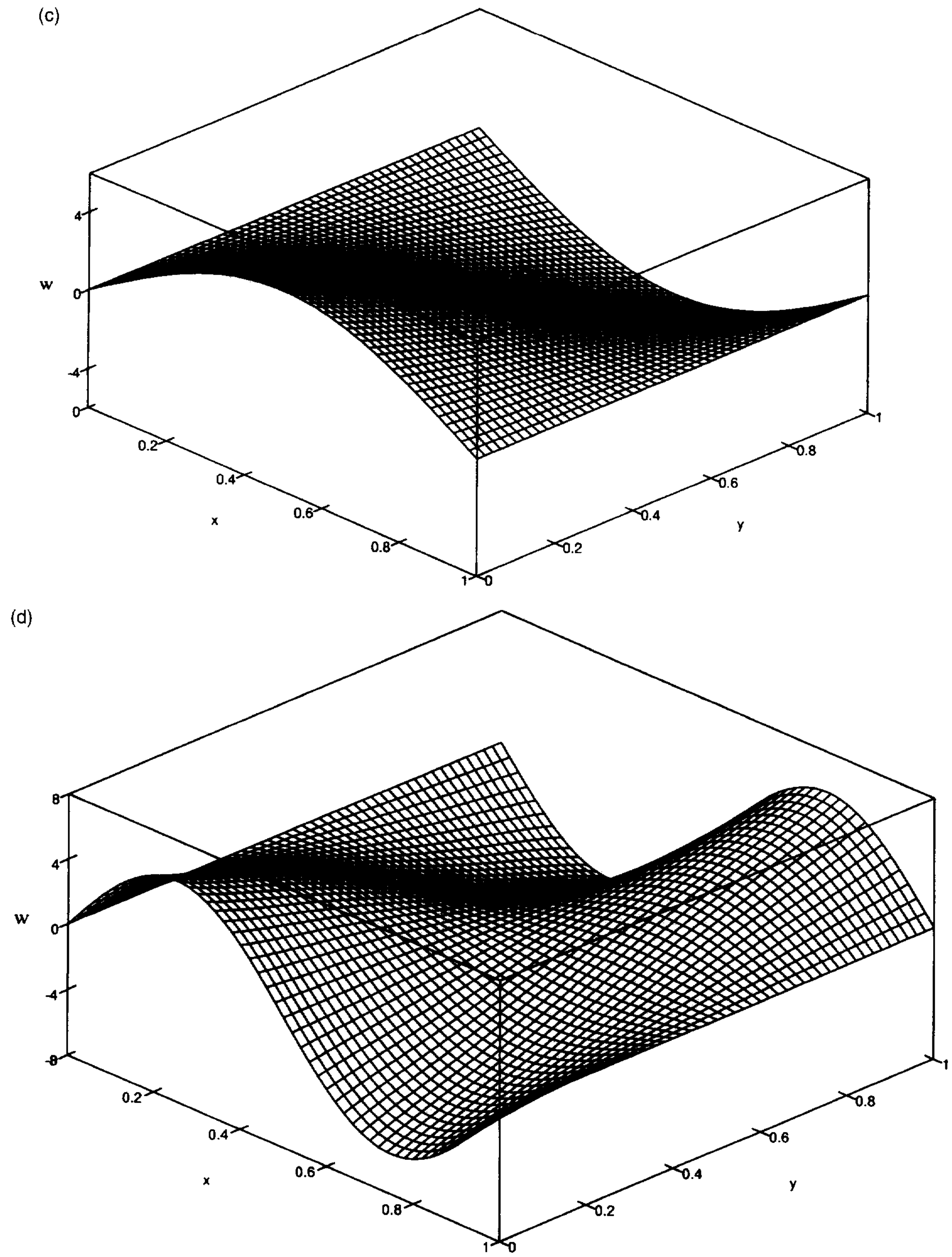

Fig. 4—*Continued.*

of w_{12} and w_{22} in the width direction are constant except near the free edges. The modes (Fig. 4(a)–(d)) are either symmetric or anti-symmetric to the midpoint of the plate.

Substitution of (16) into (8) gives

$$\lambda^2 u + 2\lambda C u_{,x} + (C^2 - 1)u_{,xx} + \varepsilon \nabla^4 u = 0. \tag{24}$$

The Galerkin solution gives an accurate approximation of the first six eigenvalues with only nine comparison functions applied. $u = w_{mk}$ satisfies (24) with $C = C_{mk}$ when the corresponding eigenvalue λ_{mk} vanishes. Therefore, the speed C_{mk}, at which non-trivial equilibrium position w_{mk} exists (static analysis), is equivalent to that at which λ_{mk} vanishes (dynamic analysis). C_{11} is the speed at which λ_{11} vanishes (the smallest circular natural frequency ω_{11} vanishes and σ_{11} impends to be non-zero). Both static and dynamic analyses predict the same speed at the onset of instability.

The eigenvalues versus axial transport speed C with $\varepsilon = 0.1$ and $\xi = 5.0$ are shown in Fig. 5. $\sigma = 0$ and the plate is stable for $C < C_1$. The natural frequencies decrease as C increases for $C < C_{11}$. σ impends to be non-zero, the smallest natural frequency vanishes, and the first mode divergent instability occurs at the speed C_{11}. There is a second stable region ($C_{21} < C < C_f$) where $\sigma = 0$. The plate experiences divergent instability for $C_{11} < C < C_{21}$. Repeated complex roots of the eigenvalues exist at the speed, C_f. The plate may experience divergent or flutter instability for $C > C_f$.

The real parts of the eigenvalues versus axial transport speed C with $\varepsilon = 0.1$ and $\xi = 1.0$ are shown in Fig. 6. There is no second stable region above C_{11}. From Fig. 2, for plates with $\xi < 1.3$, w_{12} exists at a lower transport speed than w_{21} does, because C_{12} is smaller than C_{21}. Therefore, the second stable region does not exist for plate with $\xi < 1.3$. For plates with other ε and v, a second stable region above the critical speed exists when $\xi > \xi_2(\varepsilon, v)$. Here, ξ_2 is the slenderness ratio with which $C_{12} = C_{21}$ and can be determined from (11), (12), (A.1), and (A.6). Ulsoy and Mote (1982) used plate models with *approximate* boundary conditions ($w_{,yyy} = w_{,yy} = 0$ at free edges) and led to the decoupling of the comparison functions into simple-simple and free-free beam eigenfunctions. Ulsoy's model

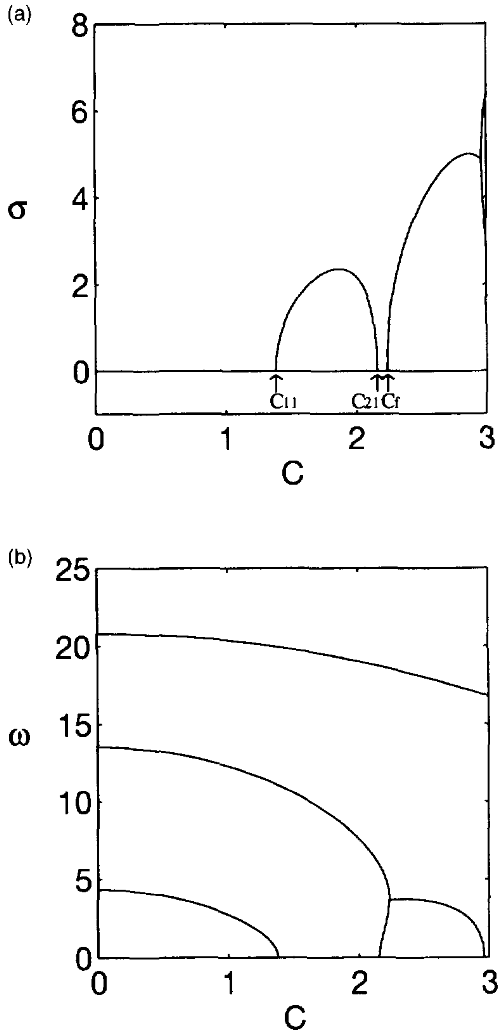

Fig. 5. The eigenvalue $\lambda = \sigma + i\omega$ vs axial transport speed C with $\varepsilon = 0.1$ and $\xi = 5.0$.

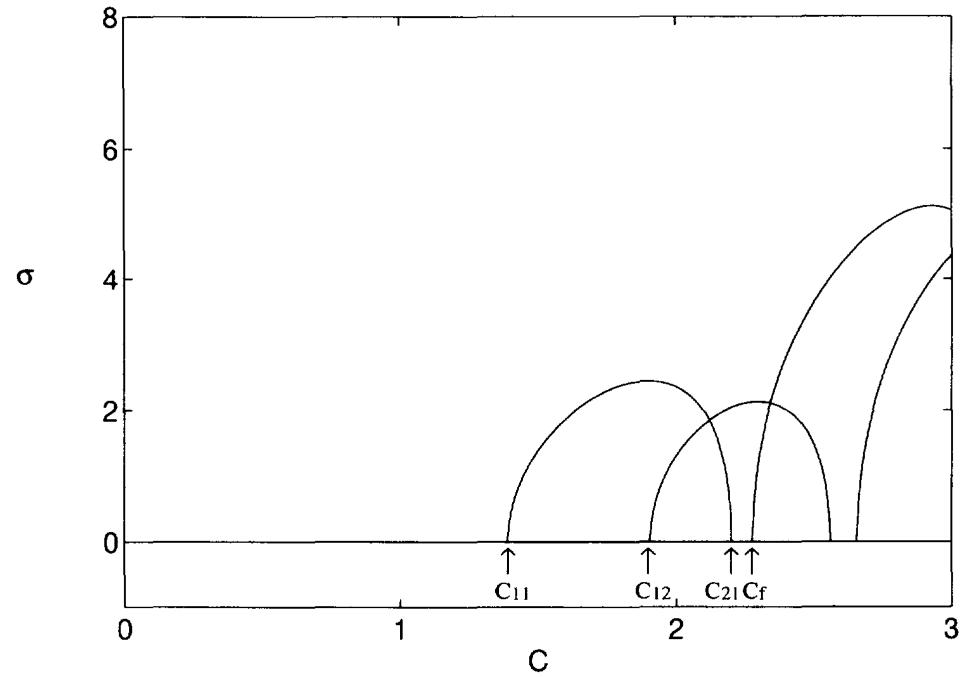

Fig. 6. The real part of the eigenvalues vs axial transport speed C with $\varepsilon = 0.1$ and $\xi = 1.0$

cannot characterize the importance of the slenderness ratio and the free edge boundary conditions on the prediction of the second stable region.

The critical speed, C_{cs}, is the speed at which the plate transport speed equals the propagation speed of a transverse wave, C_{wp}, in the plate ($C_{cs} = C_{wp}$). The wave propagation speed in a stationary plate is equivalent to that in a plate transporting along its length at constant speed. To determine the wave propagation speed in a stationary plate, let

$$\begin{aligned} w &= H(y)\sin(\pi x-\omega t)+H(y)\sin(\pi x+\omega t) \\ &= G_1\left(x-\frac{\omega}{\pi}t\right)+G_2\left(x+\frac{\omega}{\pi}t\right). \end{aligned} \tag{25}$$

G_1 represents the forward traveling wave and G_2 represents the backward traveling wave with the shape of the fundamental non-trivial equilibrium position. $H(y)$ is the function in the braces of eqn (11a) with $m = 1$. Substitution of (25) into (8) and letting $C = 0$ gives

$$-\omega^2 H+\pi^2 H+\varepsilon(\pi^4 H-2\pi^2 H_{,yy}+H_{,yyyy}) = 0. \tag{26a}$$

w_{11} in eqn (11a) satisfies (9) plus boundary conditions when the plate transports at the speed C^*. Substitution of w_{11} into (9) and letting $C = C^*$ gives

$$-(C^*)^2\pi^2 H+\pi^2 H+\varepsilon(\pi^4 H-2\pi^2 H_{,yy}+H_{,yyyy}) = 0. \tag{26b}$$

Comparison of (26a) and (26b) gives the wave propagation speed, C_{wp},

$$C_{wp} = \frac{\omega}{\pi} = C^*. \tag{27}$$

Thus,

$$C_{cs} = C^*, \tag{28}$$

the critical speed of the axially moving plate is equivalent to the speed at the onset of instability determined from the static and dynamic analyses.

CONCLUSION

Stability and vibration characteristics of two dimensional axially moving plates have been investigated. The results of this analysis are summarized as the following:

(1) The speed at the onset of instability predicted by the linear plate theory is $C^* = \sqrt{1+\varepsilon S^*}$ (eqn (13) and Fig. 3). String theory always underestimates the speed at the onset of instability. Beam theory predicts an upper bound of the speed at the onset of instability for plates. The difference between C^* and C_b^* in (23) is negligible when the stiffness ratio and the slenderness ratio of the material are small. But, the error of the speed at the onset of instability predicted by the beam theory is significant for the material with large stiffness ratio and large slenderness ratio.

(2) The speed at the onset of instability of an axially moving plate increases as the slenderness ratio decreases and the stiffness ratio increases. The smallest natural frequency decreases as the slenderness ratio increases for the plate transported at a speed less than the speed at the onset of instability.

(3) Both static and dynamic analyses predict the same speed at the onset of instability, C^*, for an axially moving plate. C^* predicted from static analysis is the speed at which the non-trivial equilibrium position w_{11} exists. C^* predicted from the dynamic analysis is the speed at which the smallest circular natural frequency ω_{11} (imaginary part of λ_{11}) vanishes and σ_{11} (real part of λ_{11}) impends to be non-zero.

(4) The speed at which the plate transport speed equals the propagation speed of a transverse wave in the plate is defined as the critical speed. The critical speed equals the speed at the onset of instability predicted by static and dynamic analyses (eqn (28)).

(5) For plates with different ε and ν, a second stable region above the critical speed exists when $\xi > \xi_2(\varepsilon, \nu)$. Here, ξ_2 is the slenderness ratio with which $C_{12} = C_{21}$ and can be determined from (11), (12), (A.1) and (A.6). All the real parts of the eigenvalues vanish in that region. This opens the possibility of stable operation at speeds higher than the critical speed.

(6) Closed form solutions of the non-trivial equilibrium positions are determined in this analysis. The deflection of the fundamental mode w_{11} is described by a sine function in the axial direction and a function including two sets of hyperbolic sines and hyperbolic cosines in the width direction. The variations of the deflections of w_{11} in the width direction are negligible except near the free edges. w_{11} is symmetric to the midpoint of the plate.

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APPENDIX

The value S_{m1} at which w_{m1} in (11a) exists can be determined from

$$Q_1Q_2Q_3Q_4[2\cosh(\alpha_m\xi)\cosh(\beta_m\xi)-2]-[(Q_1Q_4)^2+(Q_2Q_3)^2][\sinh(\alpha_m\xi)\sinh(\beta_m\xi)] = 0, \tag{A.1}$$

where

$$Q_1 = \alpha_m^3-(2-\nu)\alpha_m m^2\pi^2 \tag{A.2}$$

$$Q_2 = \beta_m^3-(2-\nu)\beta_m m^2\pi^2 \tag{A.3}$$

$$Q_3 = \alpha_m^2-\nu m^2\pi^2 \tag{A.4}$$

$$Q_4 = \beta_m^2-\nu m^2\pi^2. \tag{A.5}$$

The value S_{mn} at which w_{mn} in (11b) exists can be determined from

$$Q_1\hat{Q}_2Q_3\hat{Q}_4[2\cosh(\alpha_m\xi)\cosh(\gamma_m\xi)-2]-[(\hat{Q}_1Q_4)^2-(\hat{Q}_2Q_3)^2][\sinh(\alpha_m\xi)\sinh(\gamma_m\xi)] = 0, \tag{A.6}$$

where

$$\hat{Q}_2 = \gamma_m^3+(2-\nu)\gamma_m m^2\pi^2 \tag{A.7}$$

$$\hat{Q}_4 = \gamma_m^2+\nu m^2\pi^2. \tag{A.8}$$